Journal of Global Optimization **28:** 241–247, 2004.

Variational Inequalities: an Elementary Approach

CLAUDIO BAIOCCHI
Via Regina Margherita 43, Gavignano (Roma), Italy (e-mail: bici.nando@tiscalinet.it)

(Received and accepted 12 March 2003)

Abstract. We will deal with the following problem:
Let $\mathbf{M}$ *be an* $n \times n$ *matrix with real entries. Under which conditions the family of inequalities:*

$$\mathbf{x} \in \mathbb{R}^n; \mathbf{x} \geqslant \mathbf{0}; \mathbf{M} \cdot \mathbf{x} \geqslant \mathbf{0}$$

has non–trivial solutions?
We will prove that a sufficient condition is given by

$$m_{i,j} + m_{j,i} \geqslant 0 \quad (1 \leqslant i, j \leqslant n);$$

from this result we will derive an elementary proof of the existence theorem for Variational Inequalities in the framework of Monotone Operators.

1. The Problem

Let us firstly specify what we mean by *variational inequality for a monotone operator*. We are given $\{V, K, j, a\}$ with:

- V is a real reflexive Banach space;
- K is a closed convex non–empty subset of V;
- $j : v \to j(v)$ is a convex l.s.c. map from K to $\mathbb{R}$;
- $a : (u, v) \to a(u, v)$ is a map from $V \times V$ to $\mathbb{R}$;
- the map a is *monotone*:
  $a(u, u - v) \geqslant a(v, u - v)$ for any $u, v \in V$;
- for any $u \in V$ the map $v \to a(u, v)$ is linear continuous;
- for any $v \in V$ the map $u \to a(u, v)$ is *hemi–continuous*, say:
  $\forall w \in V$ the map $[0, 1] \ni t \to a(tu + (1 - t)w, v)$ is l.s.c.

Under these assumptions, we ask for:

$$(\mathcal{VI}) \qquad \text{Find } u \in K \text{ such that } a(u, u - v) + j(u) \leqslant j(v) \ \forall v \in K.$$

A large family of problems arising from applications can be settled into this general framework; thus many papers were devoted to the subject, and many results are known; see, e.g., [1] and the references given there. In particular, under a suitable *coerciveness* assumption (see (4) later on) one has:

$$\begin{cases}\text{the family of solutions of } (\mathcal{VI}) \\ \text{is a convex closed non–empty subset of } V\end{cases} \tag{1}$$

the uniqueness holding true e.g. if we assume a “strong” monotonicity of the map a, by adding the hypothesis:

$$\{u,v\in K, a(u,u-v)=a(v,u-v)\}\Longrightarrow\{u=v\}.$$

In order to specify the coerciveness assumption, let us recall that, in the original setting of Stampacchia [3], V is a Hilbert-space and both $a(u,\cdot), j(u)$ are linear continuous; in particular $(\mathcal{VI})$ becomes:

$$\text{find } u\in K \text{ such that } a(u,u-v)\leqslant j(u-v)\forall v\in K. \tag{2}$$

The Stampacchia’s theorem says that under the crucial assumption:

$$\exists\alpha>0 \text{ such that } a(u-v,u-v)\geqslant\alpha\|u-v\|_V^2\ \forall u,v\in K \tag{3}$$

problem (2) has a unique solution, the map $j\to u$ being lipschitz–continuous from V' to V.

We are of course dealing with a very special case of the general framework; remark that (3) implies not only the (strong) monotonicity but also a coerciveness property, that we will write in the form:

$$\begin{cases}\text{if } K \text{ is unbounded, there exists } u_0\in K \text{ such that} \\ \frac{a(v,v-u_0)+j(v)}{\|v\|}\longrightarrow 0 \text{ when } \|v\|\to\infty, v\in K\end{cases} \tag{4}$$

A crucial role in our problem will be played by the so called “Minty’s Lemma” (see [2]; later on we will prove it in a very general setting): u solves $(\mathcal{VI})$ if and only if u solves a somewhat “simpler” family of inequalities, say:

$$(\mathcal{M})\qquad \text{Find } u\in K \text{ such that } a(v,u-v)+j(u)\leqslant j(v)\ \forall v\in K.$$

Concerning problem $(\mathcal{M})$, it is quite immediate to prove that the set of solutions is a convex closed (possibly empty) subset of V. This result (that, looking directly at $(\mathcal{VI})$, is not obvious at all) joined to the coerciveness assumption, allows to greatly simplify the treatment of the problem because, as we will show, it will be sufficient to prove the existence result in the framework of:

$$K \text{ is the convex hull of some } v_1,v_2,\dots v_n\in V. \tag{5}$$

On the other hand, also in the framework of (5), the existence theorem could seem a “deep” result, the usual proof being based on a (Brower–like) fixed–point theorem.

Our goal will be to give a quite elementary proof of (1): in the framework of (5) as a first step; then, as a second step, under the general assumption (4).

2. A More Abstract Framework

We are given $\{\mathbb{V}, \mathbb{K}, F\}$ with:

$$\begin{cases} \mathbb{K} \text{ is a non-empty convex subset of a linear vector space } \mathbb{V}; \\ F : \{u, v\} \to F(u, v) \text{ is a map from } \mathbb{K} \times \mathbb{K} \text{ to } \mathbb{R}; \end{cases}$$

we ask for conditions on $\{\mathbb{V}, \mathbb{K}, F\}$ such that the family of inequalities:

$$(\mathscr{P}) \qquad u \in \mathbb{K}; \quad F(u, v) \leqslant 0 \ \forall v \in \mathbb{K}$$

has at least one solution. The change of notations with respect to the previous section (we use here $\mathbb{V}, \mathbb{K}$ instead of V, K) is motivated by the fact that we want to postpone as much as possible the use of "topological" assumptions. Remark that problem $(\mathcal{VI})$ obviously corresponds to the choice:

$$F(u, v) := a(u, u - v) + j(u) - j(v) \tag{6}$$

while, *with the same choice of* F, problem $(\mathcal{M})$ corresponds to:

$$(\mathscr{P}_*) \qquad u \in \mathbb{K}; \quad F(v, u) \geqslant 0 \ \forall v \in \mathbb{K}.$$

REMARK 1. *Let us point out that all the assumptions on* F *in the following (see (7), (8), (9) and* $F(v, v) \leqslant 0$ *in Lemma 1) are clearly satisfied with respect to the choice (6).*

The (possibly empty) family of solutions of $\mathscr{P}$ obviously coincides with the set s defined by:

$$s := \bigcap_{v \in \mathbb{K}} \sigma(v) \quad \text{where, for } v \in \mathbb{K},\ \sigma(v) := \left\{ u \in \mathbb{K} \,\middle|\, F(u, v) \leqslant 0 \right\};$$

thus we must search for assumptions sufficient to imply that such an intersection is non-empty.

Let us take a trivial example as a guideline: fix any $f : \mathbb{K} \to \mathbb{R}$ and set $F(u, v) := f(u) - f(v)$; thus the sets $\sigma(v)$ are the "level subsets" of f; and u solves $(\mathscr{P})$ if and only if u minimizes f on $\mathbb{K}$. The example clearly shows the need for some topological assumptions; on the other hand it suggests the following general strategy: if we can prove that

any $\sigma(v)$ is closed; at least one is compact

the set s will be non–empty if and only if the family $\{\sigma(v) \mid v \in \mathbb{K}\}$ satisfies the "finite intersection property".

In fact (also in order to weaken the "continuity" assumptions on F) we will first of all search for assumptions implying that problems ($\mathscr{P}$) and ($\mathscr{P}_\star$) are equivalent; then we will apply the general strategy to problem ($\mathscr{P}_\star$). In other words, setting:

$$\mathcal{S}_\star := \bigcap_{v\in\mathbb{K}} \sigma_\star(v) \quad \text{where, for } v\in\mathbb{K},\ \sigma_\star(v) := \left\{u\in\mathbb{K} \,\middle|\, F(v,u)\geqslant 0\right\}$$

we will ask for conditions implying ($\mathcal{S}\equiv\mathcal{S}_\star$ and) $\mathcal{S}\neq\emptyset$.

In our trivial example problems ($\mathscr{P}$) and ($\mathscr{P}_\star$) coincide because of the obvious property $\sigma(v)\equiv\sigma_\star(v)$ for any $v\in\mathbb{K}$; more generally, this equivalence holds true if F is "anti-symmetric": $F(u,v)+F(v,u)\equiv 0$. However in the following we will only impose the weaker assumption of "monotonicity", say:

$$F(u,v)+F(v,u)\geqslant 0\ \forall u,v\in\mathbb{K} \tag{7}$$

so that only one inclusion trivially holds true:

$\sigma(v)\subseteq\sigma_\star(v)\forall v\in\mathbb{K}$; in particular $\mathcal{S}\subseteq\mathcal{S}_\star$.

The reversed inclusion will follow from an abstract version of the Minty's Lemma, that requires some assumptions of topological nature. The first one, concerning the dependence on v of $F(u,v)$, reads:

$$\text{for any } u\in\mathbb{K} \text{ the function } v\to F(u,v) \text{ is concave s.c.s.} \tag{8}$$

Of course, concerning the semi–continuity, the topology on the linear space $\mathbb{V}$ will play an inessential role (because of the concavity); the same holds true for the second assumption of "hemi–continuity":

$$\forall u,v,w\in\mathbb{K} \text{ the map } [0,1]\ni t\to F\Big(tu+(1-t)v,w\Big) \text{ is l.s.c.} \tag{9}$$

LEMMA 1. *Let F be given with* (8), (9). *If for any $v\in\mathbb{K}$ one has $F(v,v)\leqslant 0$, then*:

$$\mathcal{S}_\star\subseteq\sigma(v)\ \forall v\in\mathbb{K};\ \textit{in particular } \mathcal{S}_\star\subseteq\mathcal{S}. \tag{10}$$

Proof. Fix any $u\in\mathcal{S}_\star$, any $v\in\mathbb{K}$ and any w in the open segment $]u,v[$. For z varying in the segment $[u,v]$ the function $z\to F(w,z)$ is positive in $z=u$ (because $u\in\mathcal{S}_\star$) and negative in the interior point $z=w$; being concave, it must be negative at the other end $z=w$ of the segment, say $F(w,v)\leqslant 0$. The inequality $F(w,v)\leqslant 0$ being satisfied for any w in the open segment $]u,v[$, from the hemi–continuity assumption we get that it $F(w,v)$ remains negative at $w=u$; say $F(u,v)\leqslant 0$, as needed. □

3. The Finite Intersection Property

From now on we will always assume that F is a monotone function that satisfies the assumptions of Lemma 1; thus we will always have $\mathcal{S}\equiv\mathcal{S}_\star$. Of course, because of the monotonicity, the assumption $F(v,v)\leqslant 0$ becomes $F(v,v)=0$; in particular the "step 0" of the finite intersection property holds true: the sets $\sigma(v)$ and $\sigma_\star(v)$ are both non–empty because each of them contains at least v. Let us prove that, more generally:

$$\text{for any choice of } v_1,v_2,\dots,v_n\in\mathbb{K}, \text{ it is } \bigcap_{j=1}^{n}\sigma_\star(v_j)\neq\emptyset; \tag{11}$$

more precisely, we will find in this intersection an element u of the convex hull of $v_1,v_2,\dots,v_n$, say $u=\sum_{j=1}^{n}\lambda_j v_j$ with $\lambda_j\geqslant 0$, $\sum_{j=1}^{n}\lambda_j=1$. In order to do that we should solve the system of inequalities

$$F(v_k,\sum_{j=1}^{n}\lambda_j v_j)\geqslant 0 \quad (k=1,2,\dots,n)$$

but, because of the concavity of $F(\cdot,v)$, it will be sufficient to solve:

$$\sum_{j=1}^{n}\lambda_j F(v_k,v_j)\geqslant 0 \quad (k=1,2,\dots,n)$$

say, denoting by $\mathbf{M}$ the $n\times n$ matrix with entries $m_{k,j}:=F(v_k,v_j)$, we must find a non–trivial solution for:

$$\mathbf{x}\in\mathbb{R}^n;\mathbf{x}\geqslant\mathbf{0};\mathbf{M}\cdot\mathbf{x}\geqslant\mathbf{0} \tag{12}$$

(then we will set $\lambda_j:=x_j/\Sigma x_k$).

We are thus faced with the problem settled out in the Abstract; concerning our matrix $\mathbf{M}$ the only property we know follows from the monotonicity of F: it is $\mathbf{M}+\mathbf{M}^*\geqslant\mathbf{0}$, where $\mathbf{M}^*$ denotes the transposed matrix of $\mathbf{M}$. Let us show that this property suffices:

THEOREM 1. *Let* $\mathbf{M}$ *be an* $n\times n$ *matrix with real entries such that* $\mathbf{M}+\mathbf{M}^*\geqslant\mathbf{0}$. *Then the system of inequalities* (12) *has non–trivial solutions.*

Proof. Let us denote by $\mathbb{P}$ the positive cone in $\mathbb{R}^n$ and by $\mathcal{C}$ the cone:

$$\mathcal{C}:=\left\{\mathbf{M}\cdot\mathbf{y}-\mathbf{z}\;\middle|\;\mathbf{y},\mathbf{z}\in\mathbb{P}\right\}$$

We then distinguish two cases:

- $\mathcal{C}$ is dense in $\mathbb{R}^n$. Then we can find two sequences $\mathbf{y}_k,\mathbf{z}_k$ in $\mathbb{P}$ such that $\mathbf{M}\cdot\mathbf{y}_k-\mathbf{z}_k$ tends to $\mathbf{1}$. Thus, for sufficiently great k, we have $\mathbf{M}\cdot\mathbf{y}_k\geqslant\mathbf{1/2}$; so that $\mathbf{y}_k$ cannot vanish and $\mathbf{M}\cdot\mathbf{y}_k\geqslant\mathbf{0}$. Being $\mathbf{y}_k\in\mathbb{P}$, the choice $\mathbf{x}:=\mathbf{y}_k$ gives a non-trivial solution of (12).

- $\mathcal{C}$ is non-dense in $\mathbb{R}^n$. Then we can find a non-zero vector $\mathbf{x}$ forming an obtuse angle with any element of $\mathcal{C}$: $(\mathbf{x}, \mathbf{M}\cdot\mathbf{y}-\mathbf{z}) \leqslant 0$ for any $\mathbf{y},\mathbf{z}\in\mathbb{P}$. The choice $\mathbf{y}=\mathbf{0}$ gives $\mathbf{x}\geqslant\mathbf{0}$, while the choice $\mathbf{z}=\mathbf{0}$ gives $\mathbf{M}^*\cdot\mathbf{x}\leqslant\mathbf{0}$. We claim that, because of $\mathbf{M}+\mathbf{M}^*\geqslant\mathbf{0}$, this $\mathbf{x}$ is a non-trivial solution of (12). In fact we already know $\mathbf{x}\neq\mathbf{0}$, $\mathbf{M}^*\cdot x\leqslant\mathbf{0}$ and $\mathbf{x}\in\mathbb{P}$; from this last relation and $\mathbf{M}+\mathbf{M}^*\geqslant\mathbf{0}$ follows $(\mathbf{M}+\mathbf{M}^*)\cdot\mathbf{x}\geqslant\mathbf{0}$; thus $\mathbf{M}\cdot\mathbf{x}\geqslant-\mathbf{M}^*\cdot\mathbf{x}\geqslant\mathbf{0}$.

Thus, in any case, non-trivial solutions of (12) do exist. □

Let us end this section with a first existence result, based on the assumption:

$$\mathbb{K} \text{ is the convex hull of a finite family } \{w_0, w_1, \ldots, w_m\}. \tag{13}$$

Remark that, in particular, we are working in a finite–dimensional space; thus $\mathbb{K}$ can be viewed as a compact convex subset of $\mathbb{R}^{m+1}$.

THEOREM 2. *Under the assumption* (13) *the family s of solutions of problem* ($\mathscr{P}$) *is a closed convex non–empty set.*

REMARK 2. *The convexity of the set s is far from being obvious; on the contrary, the convexity of $s_\star$ is quite immediate, because of the obvious convexity of each $\sigma_\star(v)$. Of course we will work with problem ($\mathscr{P}_\star$) instead of with ($\mathscr{P}$)* .

Proof. We only need to check the finite intersection property for the family (of compact sets) $\{\sigma_\star(v)\}$; and this has already been proved: see (11). □

4. The coerciveness assumption

Without some compactness assumptions the sets s and s_* could already be empty in the trivial example $F(u,v)=f(u)-f(v)$; e.g., when $\mathbb{K}=\mathbb{V}=\mathbb{R}, f(x)=\exp(-x)$. In Thm 2 the compactness was a consequence of the assumption (13); here we will assume that $\mathbb{K}, \mathbb{V}$ and F are given with:

$$\begin{cases} \mathbb{V} \text{ is a reflexive Banach space;} \\ \mathbb{K}\subseteq\mathbb{V} \text{ is closed convex and non–empty;} \\ \text{there exist a closed ball } \mathcal{B} \text{ and a } w_0\in\mathbb{K} \\ \text{ such that } F(v,w_0)>0 \text{ for any } v\in\mathbb{K}\backslash\mathcal{B}; \end{cases} \tag{14}$$

remark that the existence of the couple $(\mathcal{B}, w_0)$ could be given in the abstract setting by a boundedness of $\mathbb{K}$; while, in the framework of problem $(\mathcal{V}I)$ with the choice (6), it will obviously follow from the coerciveness assumption (4).

THEOREM 3. *Under the assumption* (14), *s is a convex closed non–empty subset of $\mathbb{V}$.*

REMARK. Here again, as in Rem. 2, in order to prove that $\mathcal{S}$ is closed and convex it is better to think in terms of the formulation $(\mathcal{P}_\star)$: our assumptions trivially imply that any $\sigma_\star(v)$ (and thus their intersection $\mathcal{S}_\star$) is a closed convex set. On the contrary, concerning the existence of at least one solution, a "lack of compactness" in problem $(\mathcal{P}_\star)$ forces to pass through the formulation in terms of problem $(\mathcal{P})$.

Proof. We firstly remark that, because of (14), one has $\sigma(w_0) \subseteq \mathcal{B}$; thus, because of $\mathcal{S}_\star = \mathcal{S} \subseteq \sigma(w_0)$, setting:

$$\sigma^\star(v) := \mathcal{B} \bigcap \sigma_*(v); \quad \mathcal{S}^\star := \bigcap_{v \in \mathbb{K}} \sigma^\star(v)$$

we have $\mathcal{S}^\star \equiv \mathcal{S}_\star$. The sets $\sigma^*(v)$ (that are convex, bounded and strongly closed) are weakly compact; thus, once again, we only need to check the finite intersection property; say we need to prove that, for any finite family $w_1, w_2, \dots, w_m \in \mathbb{K}$, there exists u_0 such that:

$$u_0 \in \mathcal{B} \bigcap \left(\bigcap_{j=1}^{m} \sigma_\star(w_j) \right). \tag{15}$$

Let $\mathbb{K}$ be defined by (13), where w_0 is the element associated to $\mathcal{B}$ in (14); from Thm 2, we know that there exists $u_0 \in \mathbb{K}$ such that $F(u_0, v) \leqslant 0 \forall v \in \mathbb{K}$; let us show that any such u_0 satisfies (15).

On one hand, from $u_0 \in \mathbb{K}$, we have $F(u_0, w_0) \leqslant 0$; thus (14) implies $u_0 \in \mathcal{B}$; on the other hand, for $j = 1, 2, \dots m$, from $F(w_j, u_0) \geqslant 0$ and the monotonicity, we get $F(u_0, w_j) \leqslant 0$, say $u_0 \in \sigma_*(w_j)$. □

References

1. Baiocchi, C. and Capelo, A. (1984), *Variational and Quasivariational Inequalities*. Whiley, Chichester.
2. Minty, J.G. (1967), On the generalization of a direct method of the calculus of variations, *Bull. AMS* 73, 315–321.
3. Stampacchia, G., Formes bilinéaires coercitives sur les ensembles convexes, *C.R.A S.*, Paris 258, 4413–4416.